Schwartz QFT Solutions

Chapter 2 – Problem 2.3

(a) Average photon energy

For a blackbody spectrum, the average photon energy is (This comes from assuming that photons are bosonic)

$$\langle E_\gamma \rangle = 2.701\, k_B T.$$

$$\langle E_\gamma \rangle \approx 2.701 \times 2.35 \times 10^{-4} \approx 6.3 \times 10^{-4}\,\text{eV}.$$

(b) Threshold proton energy

We use the Lorentz invariant

$$s = (p_p + p_\gamma)^2.$$

At threshold, the final proton and pion have no relative motion in the center-of-momentum frame, so

$$s_{\text{thr}} = (m_p + m_\pi)^2.$$

In the lab (CMB) frame, for a head-on collision,

$$s = m_p^2 + 2\epsilon(E_p + p_p),$$

where $\epsilon = E_\gamma$. For an ultra-relativistic proton, $p_p \simeq E_p$, giving

$$s \approx m_p^2 + 4\epsilon E_p.$$

Setting $s = s_{\text{thr}}$,

$$m_p^2 + 4\epsilon E_p = (m_p + m_\pi)^2,$$

so

$$E_p^{\text{thr}} = \frac{(m_p + m_\pi)^2 - m_p^2}{4\epsilon} = \frac{m_\pi(2m_p + m_\pi)}{4\epsilon}.$$

Using

$$m_p \approx 938\,\text{MeV}, \quad m_\pi \approx 135\,\text{MeV}, \quad \epsilon \approx 6.3 \times 10^{-4}\,\text{eV},$$

we obtain

$$E_p^{\text{thr}} \sim 10^{20}\,\text{eV}.$$

(c) Outgoing proton energy

At threshold, in the center-of-momentum frame the final proton and pion are produced at rest relative to each other, so they move together as a single system in the lab frame.
Thus the final state behaves like a particle of mass $m_p + m_\pi$ moving with some boost γ. The total final energy is

$$E_f = \gamma(m_p + m_\pi).$$

Since the initial proton is ultra-relativistic and the photon energy is negligible,

$$E_f \approx E_p^{\text{thr}}.$$

The energy is shared between the proton and pion in proportion to their masses:

$$E_p^{(\text{final})} = \frac{m_p}{m_p + m_\pi}\, E_p^{\text{thr}} \sim 10^{20}\,\text{eV}.$$