Schwartz QFT Solutions

Chapter 2 – Problem 2.6

(a) Using

$$k^2 - m^2 = (k^0)^2 - \vec{k}^{\,2} - m^2 = (k^0)^2 - \omega_k^2, \qquad \omega_k \equiv \sqrt{\vec{k}^{\,2} + m^2},$$

$$\delta(k^2 - m^2) = \frac{1}{2\omega_k}\Big[\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)\Big].$$

Multiplying by $\theta(k^0)$ keeps only the positive-energy root:

$$\delta(k^2 - m^2)\theta(k^0) = \frac{1}{2\omega_k}\,\delta(k^0 - \omega_k).$$

Therefore

$$\int_{-\infty}^{\infty} dk^0\,\delta(k^2 - m^2)\theta(k^0) = \int_{-\infty}^{\infty} dk^0\,\frac{1}{2\omega_k}\delta(k^0 - \omega_k) = \frac{1}{2\omega_k}.$$

(b) Let $k^\mu \mapsto k'^\mu = \Lambda^\mu{}_\nu k^\nu$ be a Lorentz transformation. Then

$$d^4k' = |\det \Lambda|\; d^4k.$$

For proper Lorentz transformations, $\det \Lambda = \pm 1$, so in particular

$$d^4k' = d^4k.$$

Thus the four-dimensional integration measure d^4k is Lorentz invariant.

(c) Consider

$$\int d^4k\,\delta(k^2 - m^2)\theta(k^0).$$

From part (b), d^4k is Lorentz invariant. Also, $k^2 = k_\mu k^\mu$ is a Lorentz scalar, so

$$\delta(k^2 - m^2)$$

is Lorentz invariant. Finally, for proper orthochronous Lorentz transformations, the sign of k^0 is preserved on the positive-energy mass shell, so $\theta(k^0)$ is also invariant there(in fact the effect of the delta function and the theta functions maps every differential element in a Lorentz invariant way). Hence the whole measure

$$d^4k\,\delta(k^2 - m^2)\theta(k^0)$$

is Lorentz invariant.
Using part (a), we may perform the k^0 integral:

$$\int d^4k\,\delta(k^2 - m^2)\theta(k^0) = \int d^3k \int_{-\infty}^{\infty} dk^0\,\delta(k^2 - m^2)\theta(k^0) = \int d^3k\,\frac{1}{2\omega_k}.$$