Schwartz QFT Solutions

Chapter 2 – Problem 2.7

We take the harmonic oscillator ladder operators to satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0,$$

and use

$$q = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \qquad p = i\sqrt{\frac{m\omega}{2}}(a^\dagger - a).$$

The coherent state in this problem is the unnormalized state

$$|z\rangle = e^{za^\dagger}|0\rangle, \qquad z \in \mathbb{C}.$$

(a)

Using the product rule,

$$\begin{aligned}\partial_z\left(e^{-za^\dagger}ae^{za^\dagger}\right) &= -a^\dagger e^{-za^\dagger}ae^{za^\dagger} + e^{-za^\dagger}aa^\dagger e^{za^\dagger} \\ &= e^{-za^\dagger}(aa^\dagger - a^\dagger a)e^{za^\dagger} \\ &= 1.\end{aligned}$$

(b)

From part (a),

$$e^{-za^\dagger}ae^{za^\dagger} = a + z.$$

Multiplying on the left by $e^{za^\dagger}$ gives

$$ae^{za^\dagger} = e^{za^\dagger}(a + z).$$

Therefore

$$\begin{aligned}a|z\rangle &= ae^{za^\dagger}|0\rangle \\ &= e^{za^\dagger}(a + z)|0\rangle \\ &= ze^{za^\dagger}|0\rangle \\ &= z|z\rangle.\end{aligned}$$

(c)

Use the number basis normalization

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \qquad \langle n| = \langle 0|\frac{a^n}{\sqrt{n!}}.$$

Then

$$\begin{aligned}\langle n|z\rangle &= \langle n|e^{za^\dagger}|0\rangle \\ &= \langle 0|\frac{a^n}{\sqrt{n!}}e^{za^\dagger}|0\rangle.\end{aligned}$$

Since $|z\rangle$ is an eigenstate of a with eigenvalue z, repeated application gives

$$a^n |z\rangle = z^n |z\rangle .$$

Therefore

$$\begin{aligned}\langle n|z\rangle &= \frac{1}{\sqrt{n!}} \langle 0| a^n |z\rangle \\ &= \frac{z^n}{\sqrt{n!}} \langle 0|z\rangle .\end{aligned}$$

But

$$\langle 0|z\rangle = \langle 0| e^{za^\dagger} |0\rangle = 1,$$

because all terms containing $a^\dagger$ are orthogonal to $\langle 0|$.

(d)

First compute the norm:

$$\langle z|z\rangle = \langle 0| e^{z^* a} e^{za^\dagger} |0\rangle .$$

Since $[z^* a, za^\dagger] = |z|^2$ is a number, the Baker–Campbell–Hausdorff formula gives (another simple very alternative is $\langle z|z\rangle = \sum_n \langle z|n\rangle \langle n|z\rangle$)

$$e^{z^* a} e^{za^\dagger} = e^{za^\dagger} e^{z^* a} e^{|z|^2}.$$

Thus

$$\langle z|z\rangle = e^{|z|^2}.$$

Also, since $a|z\rangle = z|z\rangle$ and $\langle z| a^\dagger = z^* \langle z|$,

$$\langle a\rangle = z, \qquad \left\langle a^\dagger \right\rangle = z^*,$$

where expectation values are normalized by dividing by $\langle z|z\rangle$.
Now

$$\langle q\rangle = \frac{1}{\sqrt{2m\omega}}(z + z^*), \qquad \langle p\rangle = i\sqrt{\frac{m\omega}{2}}(z^* - z).$$

Next,

$$\begin{aligned} q^2 &= \frac{1}{2m\omega}(a + a^\dagger)^2 \\ &= \frac{1}{2m\omega}\left(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2\right) \\ &= \frac{1}{2m\omega}\left(a^2 + (a^\dagger)^2 + 2a^\dagger a + 1\right). \end{aligned}$$

Therefore

$$\begin{aligned} \left\langle q^2 \right\rangle &= \frac{1}{2m\omega}\left(z^2 + (z^*)^2 + 2|z|^2 + 1\right) \\ &= \langle q\rangle^2 + \frac{1}{2m\omega}. \end{aligned}$$

So

$$\Delta q^2 = \left\langle q^2 \right\rangle - \langle q \rangle^2 = \frac{1}{2m\omega}.$$

Similarly,

$$\begin{aligned} p^2 &= -\frac{m\omega}{2}(a^\dagger - a)^2 \\ &= -\frac{m\omega}{2}\left((a^\dagger)^2 - a^\dagger a - a a^\dagger + a^2\right) \\ &= \frac{m\omega}{2}\left(2a^\dagger a + 1 - (a^\dagger)^2 - a^2\right). \end{aligned}$$

Hence

$$\begin{aligned} \left\langle p^2 \right\rangle &= \frac{m\omega}{2}\left(2|z|^2 + 1 - (z^*)^2 - z^2\right) \\ &= \langle p \rangle^2 + \frac{m\omega}{2}. \end{aligned}$$

Therefore

$$\Delta p^2 = \left\langle p^2 \right\rangle - \langle p \rangle^2 = \frac{m\omega}{2}.$$

Thus

$$\Delta q = \frac{1}{\sqrt{2m\omega}}, \qquad \Delta p = \sqrt{\frac{m\omega}{2}},$$

Thus, these coherent states are therefore minimally dispersive.

(e)

Suppose there were a normalizable eigenstate of $a^\dagger$:

$$a^\dagger |\lambda\rangle = \lambda |\lambda\rangle .$$

Write

$$|\lambda\rangle = \sum_{n=0}^{\infty} c_n |n\rangle .$$

Then

$$a^\dagger |\lambda\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n+1}\, |n+1\rangle = \lambda \sum_{n=0}^{\infty} c_n |n\rangle .$$

Comparing the coefficient of $|0\rangle$ gives

$$0 = \lambda c_0.$$

If $\lambda \neq 0$, then $c_0 = 0$, and the recursion relation then forces all $c_n = 0$.