Schwartz QFT Solutions

Chapter 3 – Problem 3.1

Consider an action of the form

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi, \partial_\nu\partial_\mu\phi).$$

Then

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta(\partial_\nu\partial_\mu\phi)\right].$$

Since variation commutes with differentiation,

$$\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi), \qquad \delta(\partial_\nu\partial_\mu\phi) = \partial_\nu\partial_\mu(\delta\phi).$$

Therefore

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\partial_\nu\partial_\mu(\delta\phi)\right].$$

Integrating the first-derivative term by parts gives

$$\int d^4x\, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) = -\int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi,$$

where boundary terms vanish. For the second-derivative term, integrate by parts twice:

$$\int d^4x\, \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\partial_\nu\partial_\mu(\delta\phi) = \int d^4x\, \partial_\mu\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\right]\delta\phi,$$

again neglecting boundary terms.
Thus

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\right)\right]\delta\phi.$$

Requiring stationarity, $\delta S = 0$, for arbitrary $\delta\phi$, gives

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\right) = 0}.$$

More generally, if the Lagrangian depends on arbitrarily high derivatives,

$$\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi, \ldots),$$

then the generalized Euler–Lagrange equation is

$$\sum_{n=0}^{\infty}(-1)^n\partial_{\mu_1}\cdots\partial_{\mu_n}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_n}\phi)}\right] = 0.$$